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Unstable oscillations in an electron gas

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With 4 figures in the text

Introduction

At the laboratory for electronics at K. Tekniska Högskolan, Stockholm, the properties of electron beams moving in a magnetic field have been studied in detail [1]. The experiments show that an energy exchange between the electrons takes place, leading to a rapid and radical change of the primary energy distribution of the electrons. The effect cannot be explained as being due to ordinary collision processes but indicates that there must exist some more effective mechanism of interaction between electrons in a magnetic field.

The main results may be summarized as follows. Electrons which are emitted in a magnetic field move in trochoidal paths and constitute a beam perpendicular to the applied electric field. At low emission, i. e. at low electron density, the motion is in accordance with what can be expected from geometrical considerations. At higher densities, however, the energy distribution of the electrons is rapidly changed in such a way that an appreciable part of the current is collected by electrodes which are negative with respect to the cathode. At the same time the beam exhibits an abnormally strong noise. It could be shown that the effect is related to the magnitude of the expression $\frac{i_e \cdot B^2}{E^{5/2}}$ where i_e is the emission current, B the magnetic field and E the electric field.

In an early attempt to give a theoretical interpretation of the observed effect [2] the author has shown that this expression has the same meaning as $\left(\frac{\omega_0}{\omega}\right)^2$ where ω_0 is the angular frequency of plasma oscillations in the electron gas and ω is the gyromagnetic frequency. A process was outlined which might give rise to oscillations which increase exponentially with time. In this paper these ideas are further developed.

During the last two years problems concerning oscillation states in electron beams have been studied by several investigators [3–9].

§ 1. In the case of ordinary plasma oscillations particles are considered, which are influenced by no forces other than those resulting from space charge fluctuations. The state of motion of the electrons is then determined by the force equation, Poisson's equation and the continuity equation. We are going to treat this simple case with the same methode as will be used later when studying the motion of electrons in a magnetic field. In both cases plane waves along one coordinate, x, are considered and furthermore it is assumed that the mean space charge of the electrons is compensated by positive ions, which can be considered as being at rest. Then the equation system takes the form:

$$\left\{\begin{aligned} &\frac{d^2x}{dt^2} = \frac{1}{m}\cdot f(x,t) && (1)\\ &\frac{\partial f}{\partial x} = 4\pi e^2\cdot n && (2)\\ &\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}\left[\frac{dx}{dt}(n_0+n)\right] = 0 && (3)\end{aligned}\right.$$

n_0 denotes the mean density of the electrons and $n_0 + n$ the actual density. If all distortions are so small that quadratic terms may be omitted this system of equations has a solution

$$f = A\cdot e^{\frac{2\pi i}{a}(x-zt)} \tag{A}$$

If $x = x_0 + \chi$ where $|\chi| \ll a$ there applies approximately:

$$\frac{d^2\chi}{dt^2} = \frac{A}{m}\cdot e^{\frac{2\pi i}{a}(x_0-zt)}$$

and integrated:

$$\frac{d\chi}{dt} = \frac{A}{m}\cdot\frac{ai}{2\pi z}\cdot e^{\frac{2\pi i}{a}(x-zt)} \tag{4}$$

After omitting quadratic terms eq. (3) may be written:

$$\frac{\partial n}{\partial t} + n_0\cdot\frac{\partial}{\partial x}\left(\frac{d\chi}{dt}\right) = 0 \tag{5}$$

which gives:

$$n = \frac{A\cdot a\cdot n_0}{2\pi m}\cdot\frac{i}{z^2}\cdot e^{\frac{2\pi i}{a}(x-zt)} \tag{6}$$

On the other hand eq. (2) gives:

$$n = \frac{A}{4\pi e^2}\cdot\frac{2\pi i}{a}\cdot e^{\frac{2\pi i}{a}(x-zt)} \tag{7}$$

A combination of the last two equations gives the characteristic equation:

$$\frac{2\pi z}{a} = \pm\sqrt{\frac{4\pi e^2 n_0}{m}} \tag{8}$$

which is the wellknown expression for the angular frequency of plasma oscillations.

§ 2. In the preceding section the electron gas has been assumed to consist of electrons performing small oscillations. A more complicated problem arises in the general case where each electron has a linear velocity. For the sake of uniformity with the problem of the following section, the motion of electrons in a magnetic field, we shall now consider a system of electrons all of which move parallel to the $x-y$-plane with the velocity v_0. Eq. (1) and (2) are still valid but instead of introducing the simple continuity eq. (3) we have to consider the system in the phase space with the coordinates x, v_x and v_y. In the initial state $(t=0)$ all electrons are contained in the space between the two cylinders $v_x^2 + v_y^2 = v_0^2$ and $v_x^2 + v_y^2 = (v_0 + \Delta v_0)^2$ and if it is assumed that all angles, φ, have the same probability the density within the region is constant, σ_0.

If, as in § 1, we try a solution:

$$f = A \cdot e^{\frac{2\pi i}{a}(x-zt)} \tag{A}$$

and put:

$$x = x_0 + v_0 \cos\varphi \cdot t + \chi$$

eq. (1) gives:

$$\delta v_x = \frac{d\chi}{dt} = \frac{A\cdot a\cdot i}{2\pi m(z - v_0\cos\varphi)} \cdot e^{\frac{2\pi i}{a}(x-zt)} \tag{4'}$$

Since

$$n_0 + n = \iint \sigma_0\, d v_x\, d v_y \tag{5'}$$

where σ_0 is a constant which does not change with time, the problem is reduced to calculating the area between the two curves in Fig. 1 after the displacement, indicated by δv_x. ($\delta v_y = 0$ in this case since there are no forces in the y-direction.)

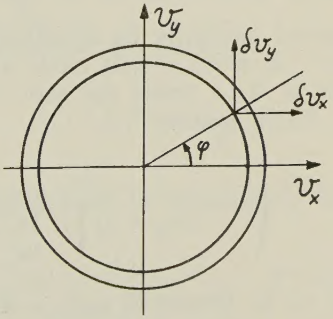

Fig. 1.

The area inside the inner curve is approximately:

$$\pi v_0^2 + v_0 \int_0^{2\pi} \delta v_x \cdot \cos\varphi \, d\varphi$$

and the area of the region between the curves is:

$$2\pi v_0 \cdot \Delta v_0 + \Delta v_0 \cdot \frac{d}{d v_0}\left(v_0 \cdot \int_0^{2\pi} \delta v_x \cos\varphi \, d\varphi\right)$$

This consideration gives:

$$n = \frac{n_0}{2\pi v_0} \cdot \frac{d}{d v_0}\left(v_0 \int_0^{2\pi} \delta v_x \cdot \cos\varphi \, d\varphi\right) \tag{6'}$$

When this expression is compared with eq. (7) the following characteristic equation is obtained:

$$-\frac{1}{\pi v_0} \cdot \frac{d}{d v_0}\left(\int_0^{\pi} \frac{\cos\varphi \, d\varphi}{\cos\varphi - \frac{z}{v_0}}\right) = \frac{1}{\omega_0^2} \cdot \left(\frac{2\pi}{a}\right)^2 \tag{8'}$$

where

$$\omega_0^2 = \frac{4\pi e^2 n_0}{m}$$

By the substitution $\operatorname{tg} \frac{\varphi}{2} = s$ the integral may be written:

$$I = \int_0^{\pi} \frac{\cos\varphi \, d\varphi}{\cos\varphi - \frac{z}{v_0}} = \int_{-\infty}^{+\infty} \frac{(s^2 - 1)\, d s}{(s^2 + 1)\left[s^2\left(1 + \frac{z}{v_0}\right) + \frac{z}{v_0} - 1\right]}$$

This integral can be evaluated explicitly by integrating from $-R$ to $+R$ along the real axis and back to $-R$ along a half circle about the origin. If $R \to \infty$ the value of the integral is $2\pi i$ times the sum of the residues at the two poles within the contour. There results:

$$I = 2\pi i\left(\frac{1}{i} - \frac{z}{2(v_0 + z)\sqrt{\frac{v_0 - z}{v_0 + z}}}\right)$$

where the square root has to be chosen in such a way that its imaginary part is positive. By inserting this expression into eq. (8′) one obtains:

$$\frac{i(v_0+z)(v_0-z)^2}{z} = \omega_0^2 \cdot \left(\frac{a}{2\pi}\right)^2 \cdot \sqrt{\frac{v_0-z}{v_0+z}} \tag{9}$$

This equation is satisfied by *real* values of z provided that $|z| > v_0$.
Then there applies:

$$\frac{1}{z}(z^2 - v_0^2)^{3/2} = \omega_0^2 \cdot \left(\frac{a}{2\pi}\right)^2$$

Among these solutions is the special case where $v_0 = 0$

$$\frac{2\pi z}{a} = \pm\,\omega_0$$

in agreement with the result of § 1.

Here we are especially interested in *complex* solutions of eq. (9). The equation may be simplified by putting.

$$z = v_0 \cdot \xi; \quad v_0 = \frac{a\,\omega_0}{2\pi} \cdot k$$

$$\frac{i(1+\xi)(1-\xi)^2}{\xi} = \frac{1}{k^2}\sqrt{\frac{1-\xi}{1+\xi}} \tag{9 b}$$

If the real and the imaginary part of ξ are separated

$$\xi = \alpha + \beta\,i = r \cdot e^{i\psi}$$

the condition that the imaginary part of the root has to be positive may be written

$$\beta^2 < \sqrt{\alpha^4 - \alpha^3 - \frac{3}{4}\alpha^2 + \frac{3}{2}\alpha + \frac{1}{4}} - \frac{1}{2} - \frac{\alpha}{2}$$

a condition which means that all solutions must be situated within the shaded part of the $\alpha - \beta$-plane in Fig. 2.

According to (9 b) the solutions must satisfy:

$$(1-\xi^2)^3 = -\frac{1}{k^4} \cdot \xi^2$$

This means that the solutions must be situated on the curve

$$r = \pm \frac{1}{\sqrt{2\cos\left(\frac{2\psi}{3} - \frac{\pi}{3}\right)}} \qquad 0 < \psi < \pi.$$

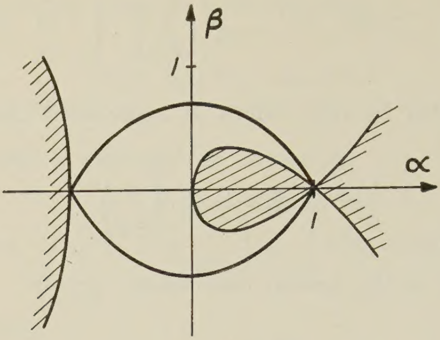

Fig. 2.

As Fig. 2 shows, this curve is in its entirety situated in the region where no solutions can exist. This means that there are no complex solutions satisfying the characteristic equation.

§ 3. We are now going to investigate the states of oscillation in a system of electrons, all of which move with the velocity v_0 in circular paths perpendicular to a magnetic field, B. The equations of motion may be written:

$$\begin{cases} \frac{d^2 x}{dt^2} = \frac{1}{m} f(x, t) - \omega \cdot \frac{dy}{dt} \\ \frac{dy}{dt} = \omega (x - x_0) \end{cases} \tag{1'}$$

where

$$\omega = \frac{eB}{m}.$$

In the phase space the initial state is the same as in § 2 and also in this case we try a solution:

$$f = A \cdot e^{\frac{2\pi i}{a}(x - zt)}. \tag{A}$$

Here we put:

$$x = x_0 + \varrho \cos(\varphi_0 + \omega t) + \chi \quad (|\chi| \ll \varrho)$$

where ϱ its the radius of curvature and χ denotes the deviation from the undisturbed circular motion.

$$\chi = e^{i\omega t} \int e^{-2i\omega t} dt \int e^{i\omega t} \cdot \frac{1}{m} f(x, t)\, dt.$$

With respect to the identity

$$e^{i\lambda \cos \psi} = \sum_{-\infty}^{-\infty} i^n \cdot J_n(\lambda) \cdot e^{in\psi}$$

where J_n denotes the Bessel function of order n with the argument λ, $f(x, t)$ can be written as a sum of integrable terms. There results:

$$\delta v_y = \omega \chi = \frac{A}{m} \cdot e^{\frac{2\pi i}{a}(x-zt)} \cdot e^{-i\lambda \cos\varphi} \cdot \sum_{-\infty}^{+\infty} \frac{i^{n-2} \cdot \omega \cdot J_n(\lambda) \cdot e^{in\varphi}}{\left(n\omega - \frac{2\pi z}{a} + \omega\right)\left(n\omega - \frac{2\pi z}{a} - \omega\right)} \qquad (4'')$$

$$\delta v_x = \frac{d\chi}{dt} = \frac{A}{m} \cdot e^{\frac{2\pi i}{a}(x-zt)} \cdot e^{-i\lambda \cos\varphi} \cdot \sum_{-\infty}^{+\infty} \frac{i^{n-1} \cdot J_n(\lambda) \cdot \left(n\omega - \frac{2\pi z}{a}\right) e^{in\varphi}}{\left(n\omega - \frac{2\pi z}{a} + \omega\right)\left(n\omega - \frac{2\pi z}{a} - \omega\right)} \qquad (4''\text{ a})$$

where

$$\varphi = \varphi_0 + \omega t \text{ and } \lambda = \frac{2\pi\varrho}{a}.$$

In conformity with the considerations in the preceding section there applies:

$$n = \frac{n_0}{2\pi v_0} \cdot \frac{d}{d v_0}\left(v_0 \int_0^{2\pi} (\delta v_x \cos\varphi + \delta v_y \sin\varphi)\, d\varphi\right)$$

or since

$$v_0 = \omega . \varrho = \frac{a\omega}{2\pi} \cdot \lambda$$

$$n = \frac{n_0}{a\omega} \cdot \frac{1}{\lambda} \cdot \frac{d}{d\lambda}\left[\lambda \int_0^{2\pi} \left(\delta v_x \cdot \frac{e^{i\varphi} + e^{-i\varphi}}{2} + \delta v_y \cdot \frac{e^{i\varphi} - e^{-i\varphi}}{2i}\right) d\varphi\right].$$

When the expressions for δv_x and δv_y are inserted there results after integration:

$$n = A \cdot e^{\frac{2\pi i}{a}(x-zt)} \cdot \frac{\pi n_0}{a\omega m} \cdot \frac{1}{\lambda} \frac{d}{d\lambda}\left(\lambda \cdot \sum_{-\infty}^{+\infty} \frac{J_n (J_{n+1} - J_{n-1})}{n\omega - \frac{2\pi z}{a}}\right).$$

A comparison with eq. (7) gives the characteristic equation:

$$\frac{1}{\lambda} \cdot \frac{d}{d\lambda}\left(\lambda \cdot \sum_{-\infty}^{+\infty} \frac{J_n (J_{n+1} - J_{n-1})}{n - \frac{2\pi z}{a\omega}}\right) = 2i \cdot \left(\frac{\omega}{\omega_0}\right)^2. \qquad (8'')$$

If we put:

$$\left(\frac{\omega_0}{\omega}\right)^2 = \frac{4\pi m n_0}{B^2} = K \text{ and } \frac{2\pi z}{a\,\omega} = \xi$$

this equation may be written:

$$\frac{J_1^2 - J_0^2}{\xi^2} + \sum_{+1}^{+\infty} \frac{J_{n+1}^2 - 2\,J_n^2 + J_{n-1}^2}{\xi^2 - n^2} = \frac{i}{K \cdot \xi}. \tag{9''}$$

If ξ is divided into its real and imaginary parts, $\xi = \alpha + \beta i$, the following system of equations is obtained

$$\left\{\begin{aligned} &\frac{J_1^2 - J_0^2}{(\alpha^2 - \beta^2)^2 + 4\,\alpha^2\beta^2} + \frac{J_2^2 - 2\,J_1^2 + J_0^2}{(\alpha^2 - \beta^2 - 1)^2 + 4\,\alpha^2\beta^2} + \\ &\qquad + \frac{J_3^2 - 2\,J_2^2 + J_1^2}{(\alpha^2 - \beta^2 - 4)^2 + 4\,\alpha^2\beta^2} + \cdots = \frac{-1}{2\,K \cdot \beta \cdot (\alpha^2 + \beta^2)} \\ &\frac{J_2^2 - 2\,J_1^2 + J_0^2}{(\alpha^2 - \beta^2 - 1)^2 + 4\,\alpha^2\beta^2} + \frac{4\,(J_3^2 - 2\,J_2^2 + J_1^2)}{(\alpha^2 - \beta^2 - 4)^2 + 4\,\alpha^2\beta^2} + \\ &\qquad + \frac{9\,(J_4^2 - 2\,J_3^2 + J_2^2)}{(\alpha^2 - \beta^2 - 9)^2 + 4\,\alpha^2\beta^2} + \cdots = \frac{-1}{2\,K \cdot \beta} \end{aligned}\right.$$

With given values of the parameters λ and K, the problem is to find values of α and β satisfying these equations.

Let us for a moment return to the solution (A)

$$\begin{aligned} f &= A \cdot e^{\frac{2\pi i}{a}(x - zt)} \\ &= A \cdot e^{\beta\omega t} \cdot e^{\left(\frac{2\pi i x}{a} - i\alpha\omega t\right)}. \end{aligned}$$

If the equation system can be satisfied by positive values of β, this would mean that there exist states of oscillation, with an amplitude which increases exponentially with time. A numerical treatment of the equations shows that such solutions exist.

The full curves of Fig. 3 represent a system of solutions calculated with λ-values between 1.2 and 2.8. The dotted curves indicate a similar system with λ-values in the range 2.2 to 4.2.

At a low value of K, i. e. at low electron density, there exist unstable oscillations with the frequency within small intervals about multiples of the gyromagnetic frequency. When the electron density is increased the intervals are successively broadened and when the density is so high that $K \simeq 10$ unstable oscillations exist at all frequencies. At such high densities there exist solutions even when $\alpha = 0$, which means that also stationary distortions may be amplified.

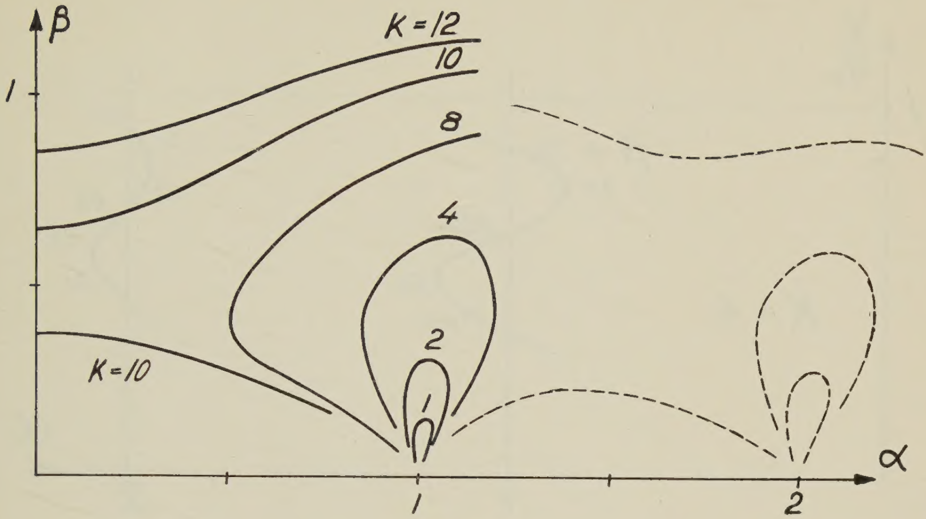

Fig. 3.

As Fig. 3 shows, β can reach values as high as unity. In such a case a distortion is amplified ~ 500 times during one single revolution of the electrons. It should be remembered that the calculations are valid only when the distortions are so small that the problem may be considered as linear. Thus, the equations and diagrams given above will only describe the initial phase of the growth of the distortions but even without knowledge of the whole process it may be concluded that an electron gas with ordered electron motion is unstable and attains its stationary state very much faster than would be the case if only collision processes were to be considered.

§ 4. Fig. 4 gives a diagram which illustrates the propagation of plane waves in the electron gas. If V denotes the phase velocity there applies:

$$\frac{V}{v_0} = \frac{\alpha}{\lambda}.$$

By keeping K at a constant value V can be given as a function of α. The whole curves represent increasing waves and the dotted lines represent ordinarily damped waves, which in this case are of secondary importance.

§ 5. In the preceding sections it was shown that a system of electrons, all of which move with the same velocity perpendicular to a magnetic field, is unstable. Small primary distortions, e. g. statistical density fluctuations are rapidly amplified and give rise to large density variations accompanied by electric fields within the electron gas. Most probably a process of this kind is the reason for the noise which is observed in the trochotrons. Furthermore it seems reasonable that the electrons are accelerated or retarded in the electric fields by a process similar to that in the cyclotron whereby the energy distribution curve of the electrons is broadened.

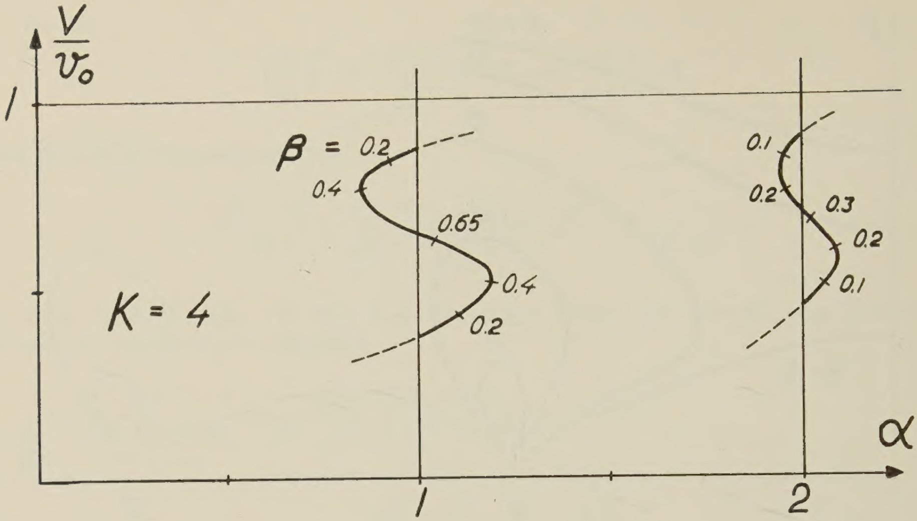

Fig. 4.

A process of this kind must be accompanied by the emission of electromagnetic radiation. The conditions in the inner solar corona agree fairly well with those in the trochotrons. The electron density, the electron velocity and the magnetic field are of the same order of magnitude in both cases. Thus it seems possible that a process of the kind outlined here may be the source of the observed solar noise.

As long as the magnetic field is constant, the electron gas is in a state of equilibrium with a low noise level. If the magnetic field varies, e. g. in connection with the occurrence of sunspots, the equilibrium is disturbed and the occurence of unstable oscillations increases the noise level.

The emission of galactic noise may possibly be interpreted as being due to some similar mechanism.

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